

On Isotopic Characterization of Central Loops ^{*†}

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Abstract

The representation sets of central loops are investigated and the results obtained are used to construct a finite C-loop. It is shown that for certain types of isotopisms, the central identities are isotopic invariant.

1 Introduction

The isotopic invariance or the universality of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [13] and [12] in the 1960s and later on in this 21st century by Phillips and Vojtěchovský [26], [27] and [21] have been of interest to researchers in loop theory in the recent past. Falconer [10] and [11] investigated isotopy invariants in quasigroups and loops. Loops such as Bol loops, Moufang loops, extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. But for LC-loops, RC-loops and C-loops, up till this moment, there is no outstanding result on their isotopic invariance.

Bol-Moufang type of quasigroups(loops) are not the only quasigroups(loops) whose universality have been considered. Some others are flexible loops, F-quasigroups, totally symmetric quasigroups, distributive quasigroups, weak inverse property loops(WIPLs), cross inverse property loops(CIPLs), semi-automorphic inverse property loops(SAIPLs) and inverse property loops(IPLs). As shown in Bruck [25], a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop, so an IPL is universal if and only if it is a Moufang loop. Recently, Kepka et. al. [17], [18], [19] solved the Belousov problem concerning the universality of F-quasigroup which has been open since 1967. The universality of WIPLs and CIPLs have been addressed by OSborn [24] and Artzy [1] respectively while the

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universality of elasticity(flexibility) was studied by Syrbu [30]. Basarab [3] later continued the work of Osborn on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs([2]). The universality of SAIPs and Osborn loops is still an open problem to be solved as stated by Kinyon during the LOOPS'99 conference and Milehigh conference 2005([16]) respectively. After the consideration of universal AIPLs by Karklinsh and Klin [15], Basarab [4] obtained a sufficient condition for which a universal AIPL is a G-loop.

LC-loops, RC-loops and C-loops are loops that satisfy the identities

$$(xx)(yz) = (x(xy))z, (zy)(xx) = z((yx)x) \text{ and } x(y(yz)) = ((xy)y)z \text{ respectively.}$$

These three types of loops shall be collectively called central loops. In the theory of loops, central loops are some of the least studied loops. They have been studied by Phillips and Vojtěchovský [26], [27], [28], Kinyon et. al. [22], [20], [21], Ramamurthi and Solarin [29], Fenyves [13] and Beg [5], [6]. The difficulty in studying them is as a result of the nature of the identities defining them when compared with other Bol-Moufang identities. It can be noticed that in the aforementioned LC identity, the two x variables are consecutively positioned and neither y nor z is between them. A similarly observation is true in the other two identities(i.e the RC and C identities). But this observation is not true in the identities defining Bol loops, Moufang loops and extra loops. Fenyves [13] gave three equivalent identities that define LC-loops, three equivalent identities that define RC-loops and only one identity that defines C-loops. But recently, Phillips and Vojtěchovský [26], [27] gave four equivalent identities that define LC-loops and four equivalent identities that define RC-loops. Three of the four identities given by Phillips and Vojtěchovský are the same as the three already given by Fenyves.

Their basic properties are found in [28], [29], [13] and [9]. The left(right) representation of a loop L denoted by $\Pi_\lambda(\Pi_\rho)$ is the set of all left(right) translation maps on the loop i.e if L is a loop, then

$$\Pi_\lambda = \{L_x : L \rightarrow L \mid x \in L\} \text{ and } \Pi_\rho = \{R_x : L \rightarrow L \mid x \in L\}$$

where $R_x : L \rightarrow L$ and $L_x : L \rightarrow L$, defined as $yR_x = yx$ and $yL_x = xy$ respectively $\forall x, y \in L$ are bijections. L is said to be a central square loop if $x^2 \in Z(L, \cdot) \forall x \in L$ where $Z(L, \cdot)$ is the center of L . L is said to be left alternative if $\forall x, y \in L, x \cdot xy = x^2y$ and is said to be right alternative if $\forall x, y \in L, yx \cdot x = yx^2$. Thus, L is said to be alternative if it is both left and right alternative. The set $S(L, \cdot)$ of all bijections in a loop (L, \cdot) forms a group called the permutation group of the loop (L, \cdot) . The triple (U, V, W) such that $U, V, W \in S(L, \cdot)$ is called an autotopism of L if and only if

$$xU \cdot yV = (x \cdot y)W \forall x, y \in L.$$

The group of autotopisms of L is denoted by $Aut(L, \cdot)$. If $U \in S(L, \cdot)$ such that $(U, U, U) \in Aut(L, \cdot)$, then U is called an automorphism. Let (L, \cdot) and (G, \circ) be two distinct loops. The triple

$$(U, V, W) : (L, \cdot) \rightarrow (G, \circ) \text{ such that } U, V, W : L \rightarrow G$$

are bijections is called a loop isotopism if and only if

$$xU \circ yV = (x \cdot y)W \quad \forall x, y \in L.$$

The representation sets of central loops are investigated and the results obtained are used to construct a finite C-loop. We show that for certain types of isotopisms, LC-identity and RC-identity are isotopic invariant. For C-identity, this is true for commutative loops in general. If the loops are non-commutative, then they must be alternative central square for the C-identity to be an invariant property. Under these types of isotopisms, alternative central square loop isotopes of Moufang loops with 'lack of commutativity property', some types of groups and RA-loops are found to be C-loops.

2 Preliminaries

Definition 2.1 ([25], III.3.9 Definition, III.3.10 Definition, III.3.15 Definition)

Let (L, \cdot) be a loop and $U, V, W \in S(L, \cdot)$.

1. If $(U, V, W) \in \text{Aut}(L, \cdot)$ for some V, W , then U is called *autotopic*,
 - the set of autotopic bijections in a loop (L, \cdot) is represented by $\Sigma(L, \cdot)$.
2. If $(U, V, W) \in \text{Aut}(L, \cdot)$ such that $W = U, V = I$, then U is called λ -regular,
 - the set of all λ -regular bijections in a loop (L, \cdot) is represented by $\Lambda(L, \cdot)$.
3. If $(U, V, W) \in \text{Aut}(L, \cdot)$ such that $U = I, W = V$, then V is called ρ -regular,
 - the set of all ρ -regular bijections in a loop (L, \cdot) is represented by $P(L, \cdot)$.
4. If $\exists V \in S(L, \cdot)$ such that $xU \cdot y = x \cdot yV \quad \forall x, y \in L$, then U is called μ -regular while $U' = V$ is called its *adjoint*.
 - The set of all μ -regular bijections in a loop (L, \cdot) is denoted by $\Phi(L, \cdot)$, while the collection of all adjoints in the loop (L, \cdot) is denoted by $\Phi^*(L, \cdot)$.

The followings results will be judiciously used in this study.

Theorem 2.1 ([25], III.3.4 Theorem)

If two quasigroups are isotopic then their groups of autotopisms are isomorphic.

Theorem 2.2 ([25], III.3.11 Theorem, III.3.16 Theorem)

The set $\Lambda(Q, \cdot) \left(P(Q, \cdot) \right) \left[\Phi(Q, \cdot) \right]$ of all λ -regular (ρ -regular) [μ -regular] bijections of a quasigroup (Q, \cdot) is a subgroup of the group $\Sigma(Q, \cdot)$ of all autotopic bijections of (Q, \cdot) .

Corollary 2.1 ([25], III.3.12 Corollary, III.3.16 Theorem)

If two quasigroups Q and Q' are isotopic, then the corresponding groups Λ and Λ' (P and P') [Φ and Φ'] [Φ^* and Φ'^*] are isomorphic.

3 Main Results

3.1 The Representation sets of Central Loops

Theorem 3.1 *Let $\Pi_\lambda(\Pi_\rho)$ be the left(right) representation of a loop L . L is a LC(RC)-loop $\Leftrightarrow \alpha, \beta \in \Pi_\lambda(\Pi_\rho) \Rightarrow \alpha\beta^2 \in \Pi_\lambda(\Pi_\rho)$.*

Proof

Let L be an LC-loop, then $(x \cdot xy)z = x(x \cdot yz)$. Using the definition of L_x , we have $L_{x \cdot xy} = L_y L_x^2$. Replacing $x \cdot xy$ in L and making $\alpha = L_y$ and $\beta = L_x$, $\alpha\beta^2 \in \Pi_\lambda$.

Conversely, do the reverse of the above. For Π_ρ when L is an RC-loop, $z(yx \cdot x) = (zy \cdot x)x$. The proof goes in the same manner by using R_x .

Theorem 3.2 *Let $\Pi_\lambda(\Pi_\rho)$ be the left(right) representation of a loop L . L is a C-loop $\Leftrightarrow \alpha, \beta \in \Pi_\lambda(\Pi_\rho) \Rightarrow \alpha\beta^2, \alpha^2\beta \in \Pi_\lambda(\Pi_\rho)$.*

Proof

Let L be a C-loop, then $(yx \cdot x)z = y(x \cdot xz) \Rightarrow L_{yx \cdot x} = L_x^2 L_y$. Let $\alpha = L_x, \beta = L_y$, then replacing $yx \cdot x$ in L , $\alpha^2\beta \in \Pi_\lambda$. L is a C-loop $\Leftrightarrow L$ is an RC-loop and LC-loop by [13], hence by Theorem 3.1, $\alpha\beta^2, \alpha^2\beta \in \Pi_\lambda$.

Conversely let $\alpha, \beta \in \Pi_\lambda \Rightarrow \alpha\beta^2, \alpha^2\beta \in \Pi_\lambda$. Take $\alpha = L_x, \beta = L_y$ then $(yx \cdot x)z = y(x \cdot xz) \Rightarrow L$ is a C-loop. For Π_ρ the procedure is similar with R_x .

Theorem 3.3 *If $\Pi_\lambda(\Pi_\rho)$ is the left(right) representation of a LC(RC)-loop and $\beta \in \Pi_\lambda(\Pi_\rho)$ then $\beta^n \in \Pi_\lambda(\Pi_\rho) \forall n \in \mathbb{Z}$.*

Proof

By Theorem 3.1, $\alpha, \beta \in \Pi_\lambda(\Pi_\rho) \Rightarrow \alpha\beta^2 \in \Pi_\lambda(\Pi_\rho)$. Using induction, when $\alpha = I$, $\beta^2 \in \Pi_\lambda(\Pi_\rho)$, when $\alpha = \beta$, $\beta^3 \in \Pi_\lambda(\Pi_\rho)$ and when $\alpha = \beta^k$, $\beta^{k+2} \in \Pi_\lambda(\Pi_\rho)$ hence, $\beta^n \in \Pi_\lambda(\Pi_\rho) \forall n \in \mathbb{Z}^+$. If L is an LC(RC)-loop then it is a left(right) inverse property loop by [13]. Hence $\beta = L_{x^{-1}}(R_{x^{-1}}) \in \Pi_\lambda(\Pi_\rho) \Rightarrow \beta = L_x^{-1}(R_x^{-1}) \in \Pi_\lambda(\Pi_\rho) \forall x \in L$. By the earlier result, $\beta^n \in \Pi_\lambda(\Pi_\rho) \forall n \in \mathbb{Z}^-$. Whence $\beta^n \in \Pi_\lambda(\Pi_\rho) \forall n \in \mathbb{Z}$.

Corollary 3.1 *If $\Pi_\lambda(\Pi_\rho)$ is the left(right) representation of a C-loop and $\alpha \in \Pi_\lambda(\Pi_\rho)$ then $\alpha^n \in \Pi_\lambda(\Pi_\rho) \forall n \in \mathbb{Z}$.*

Proof

Using Theorem 3.2, $\alpha, \beta \in \Pi_\lambda(\Pi_\rho) \Rightarrow \alpha\beta^2, \alpha^2\beta \in \Pi_\lambda(\Pi_\rho)$. The rest of the proof is similar to that in Theorem 3.3.

3.2 Isotopes of LC, RC and C-loops

Throughout this subsection, the following notations for translations will be adopted; $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$ for a loop while $L'_x : y \mapsto xy$ and $R'_x : y \mapsto yx$ for its loop isotope.

Theorem 3.4 *A loop L is an LC-loop $\Leftrightarrow (L_x^2, I, L_x^2) \in \text{Aut}(L) \forall x \in L$.*

Proof

Let L be an LC-loop $\Leftrightarrow (x \cdot xy)z = (xx)(yz) \Leftrightarrow (x \cdot xy)z = x(x \cdot yz)$ by [9] $\Leftrightarrow (L_x^2, I, L_x^2) \in \text{Aut}(L) \forall x \in L$.

Theorem 3.5 *A loop L is an RC-loop $\Leftrightarrow (I, R_x^2, R_x^2) \in \text{Aut}(L) \forall x \in L$.*

Proof

Let L be an RC-loop, then $z(yx \cdot x) = zy \cdot xx \Leftrightarrow y(yx \cdot x) = (zy \cdot x)x$ by [9] $\Leftrightarrow (I, R_x^2, R_x^2) \in \text{Aut}(L) \forall x \in L$.

Lemma 3.1 *A loop is an LC(RC)-loop $\Leftrightarrow L_x^2(R_x^2)$ is $\lambda(\rho)$ -regular i.e $L_x^2(R_x^2) \in \Lambda(L)(P(L))$.*

Proof

Using Theorem 3.4(Theorem 3.5), the rest follows from the definition of $\lambda(\rho)$ -regular bijection.

Theorem 3.6 *A loop L is a C-loop $\Leftrightarrow R_x^2$ is μ -regular and the adjoint of R_x^2 , denoted by $(R_x^2)^* = L_x^2$ i.e $R_x^2 \in \Phi(L)$ and $L_x^2 \in \Phi^*(L)$.*

Proof

Let L be a C-loop then $(yx \cdot x)z = y(x \cdot xz) \Rightarrow yR_x^2 \cdot z = y \cdot zL_x^2 \Rightarrow R_x^2 \in \Phi(L)$ and $L_x^2 \in \Phi^*(L)$. Conversely: do the reverse of the above.

Theorem 3.7 *Let (G, \cdot) and (H, \circ) be any two distinct loops. If the triple $\alpha = (A, B, B)$ ($\alpha = (A, B, A)$) is an isotopism of (G, \cdot) upon (H, \circ) , then (G, \cdot) is an LC(RC)-loop $\Leftrightarrow (H, \circ)$ is a LC(RC)-loop.*

Proof

By Lemma 3.1, G is an LC(RC)-loop $\Leftrightarrow L_x^2(R_x^2) \in \Lambda(G)(P(G))$. Using the result in [7], for each case ; $L'_{xA} = B^{-1}L_xB$ and $R'_{xB} = A^{-1}R_xA \forall x \in G$. By Corollary 2.1, there exists isomorphisms $\Lambda(G) \rightarrow \Lambda(H)$ and $P(G) \rightarrow P(H)$. Thus $L_y'^2(R_y'^2) \in \Lambda(H)(P(H)) \Leftrightarrow H$ is an LC(RC)-loop.

Theorem 3.8 *Let (G, \cdot) and (H, \circ) be two distinct loops. If G is a central square C-loop, H an alternative central square loop and the triple $\alpha = (A, B, B)$ ($\alpha = (A, B, A)$) is an isotopism of G upon H , then H is a C-loop.*

Proof

G is a C-loop $\Leftrightarrow R_x^2 \in \Phi(G)$ and $(R_x^2)^* = L_x^2 \in \Phi^*(G)$ for all $x \in G$ and using the result in [7] ; $L'_{xA} = B^{-1}L_xB$ and $R'_{xB} = A^{-1}R_xA \forall x \in G$. Using Corollary 2.1, $R_y'^2 \in \Phi(H)$ and $(R_y'^2)^* = L_y'^2 \in \Phi^*(H) \Leftrightarrow H$ is a C-loop.

Theorem 3.9 *Let (G, \cdot) and (H, \circ) be commutative loops. If $\alpha = (A, B, B)$ or $\alpha = (A, B, A)$ is an isotopism of G upon H , then G is a C-loop if and only if H is a C-loop.*

Proof

The proof is similar to that of Theorem 3.8

Corollary 3.2 *Under a triple of the form $\alpha = (A, B, A)$ or $\alpha = (A, B, B)$, if a Dihedral group or Quaternion group(Q_8) or Cayley loop or $C_2 \times C_2$ or $C_2 \times C_2 \times C_2$ or RA-loops or $Q_8 \times E \times A$ or $M_{16}(Q_8) \times E \times A$ where E is an elementary abelian 2-group, A is an abelian group(all of whose elements have finite odd order) and $M_{16}(Q_8)$ is a Cayley loop, is isotopic to an alternative central square loop G , then G is a C-loop.*

Proof

From [14] and [8], the Dihedral group D_4 of order 8, Quaternion group Q_8 of order 8, Cayley loop, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$ and ring alternative loops(RA-loops) are all central square. Hence, by Theorem 3.8, the claim that G is a C-loop follows. From [23] and the fact that Q_8 is an Hamiltonian Moufang loop, $Q_8 \times E \times A$ and $M_{16}(Q_8) \times E \times A$ are both central square. Hence, by Theorem 3.8, the claim that G is a C-loop follows.

Construction Let Π_ρ be the right representation of a C-loop of order 12. If $\alpha, \beta, \gamma \in \Pi_\rho$ are given by :

$$\alpha = (0 \ 10 \ 1 \ 11 \ 2 \ 9)(3 \ 7 \ 4 \ 8 \ 5 \ 6) = R_{10},$$

$$\beta = (0 \ 3)(1 \ 4)(2 \ 5)(6 \ 10)(7 \ 11)(8 \ 9) = R_3,$$

$$\gamma = (0 \ 7 \ 2 \ 6 \ 1 \ 8)(3 \ 10 \ 5 \ 9 \ 4 \ 11) = R_7.$$

Then with Theorem 3.2 and Corollary 3.1, $\alpha, \beta, \gamma \in \Pi_\rho$ generate other members of Π_ρ by considering the multiplications $\alpha\beta^2, \alpha^2\beta, \gamma^n \in \Pi_\rho \ \forall n \in \mathbb{Z}$. These give us a finite C-loop whose bordered multiplication table is shown by Table 1 on Page 7.

References

- [1] R. Artzy (1959), *Crossed inverse and related loops*, Trans. Amer. Math. Soc. 91, 3, 480–492.
- [2] A. S. Basarab (1967), *A class of WIP-loops*, Mat. Issled. 2(2), 3-24.
- [3] A. S. Basarab(1970), *Isotopy of WIP loops*, Mat. Issled. 5, 2(16), 3-12.
- [4] A. S. Basarab and A. I. Belioglo (1979), *Universal automorphic inverse G-loops*, Quasi-groups and loops, Mat. Issled. 71, 3–7.
- [5] A. Beg (1977), *A theorem on C-loops*, Kyungpook Math. J. 17(1), 91–94.
- [6] A. Beg (1980), *On LC-, RC-, and C-loops*, Kyungpook Math. J. 20(2), 211–215.

- [7] R. Capodaglio Di Cocco (1993), *On Isotopism and Pseudo-Automorphism of the loops*, Est.da : Boll. Uni. Mat. Italiana[7-A], 199–205.
- [8] O. Chein and E. G. Goodaire (2002), *Three-Generator Indecomposable RA Loops*, Comm. Alg. 30, 3559–3564.
- [9] J. Dénes and A. D. Keedwell (1974), *Latin square and their applications*, Academic Press, New York, London.
- [10] E. Falconer (1969), *Quasigroup identities invariant under isotopy*, Ph.D thesis, Emory University.
- [11] E. Falconer (1970), *Isotopy invariants in quasigroups*, Trans. Amer. Math. Soc. 151, 2, 511–526.
- [12] F. Fenyves (1968), *Extra loops I*, Publ. Math. Debrecen, 15, 235–238.
- [13] F. Fenyves (1969), *Extra Loops II*, Publ. Math. Debrecen 16, 187–192.
- [14] E. G. Goodaire, E. Jespers and C. P. Milies (1996), *Alternative Loop Rings*, NHMS(184) Elsevier.
- [15] B. B. Karklinsh and V. B. Klin (1976), *Universal automorphic inverse property loops*, Collections: sets and quasigroups, Shtintsa.
- [16] M. K. Kinyon (2005), *A survey of Osborn loops*, A Talk at the Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado.

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	5	3	1	2	0	10	11	9	7	8	6
5	5	3	4	2	0	1	11	9	10	8	6	7
6	6	7	8	10	11	9	0	1	2	5	3	4
7	7	8	6	11	9	10	1	2	0	3	4	5
8	8	6	7	9	10	11	2	0	1	4	5	3
9	9	10	11	8	6	7	3	4	5	2	0	1
10	10	11	9	6	7	8	4	5	3	0	1	2
11	11	9	10	7	8	6	5	3	4	1	2	0

Table 1: A non-associative C-loop of order 12

- [17] T. Kepka, M. K. Kinyon, J. D. Phillips, *The structure of F -quasigroups*, communicated for publication.
- [18] T. Kepka, M. K. Kinyon, J. D. Phillips, *F -quasigroups and generalised modules*, communicated for publication.
- [19] T. Kepka, M. K. Kinyon, J. D. Phillips, *F -quasigroups isotopic to groups*, communicated for publication.
- [20] M. K. Kinyon, K. Kunen, J. D. Phillips (2002), *A generalization of Moufang and Steiner loops*, Alg. Univer. 48,1, 81–101.
- [21] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský (2005), *Loops of Bol-Moufang type with a subgroup of index two*, Bul. Acad. Stiinte Repub. Mold. Mat. 3(49), 71–87.
- [22] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský, *C-loops : Extensions and construction*, J. Alg. and Applica. (to appear).
- [23] D. A. Norton (1952), *Hamiltonian loops*, Proc. Amer. Math. Soc.3, 56–65.
- [24] J. M. Osborn (1961), *Loops with the weak inverse property*, Pac. J. Math. 10, 295–304.
- [25] H. O. Pflugfelder (1990), *Quasigroups and Loops : Introduction*, Heldermann Verlag, Sigma series in Pure Mathematics : 7.
- [26] J. D. Phillips and P. Vojtěchovský (2005), *The varieties of loops of Bol-Moufang type*, Alg. Univer. 3(54), 259–383..
- [27] J. D. Phillips and P. Vojtěchovský (2005), *The varieties of quasigroups of Bol-Moufang type : An equational approach*, J. Alg. 293, 17–33.
- [28] J. D. Phillips and P. Vojtěchovský (2006), *On C-loops*, Publ. Math. Debrecen. 68, 1-2, 115–137.
- [29] V. S. Ramamurthi and A. R. T. Solarin (1988), *On finite right central loops*, Publ. Math. Debrecen, 35, 260–264.
- [30] P. N. Syrbu (1996), *On loops with universal elasticity*, Quasigroups and Related Systems, 3, 41–54.